

# DIVERGENCE OF GENERAL LOCALIZED OPERATORS ON THE SETS OF MEASURE ZERO

G. A. KARAGULYAN

**ABSTRACT.** We consider sequences of linear operators  $U_n f(x)$  with localization property. It is proved that for any set  $E$  of measure zero there exists a set  $G$  for which  $U_n \mathbb{I}_G(x)$  diverges at each point  $x \in E$ . This result is a generalization of analogous theorems known for the Fourier sums operators with respect to different orthogonal systems.

## 1. INTRODUCTION

In 1876 P. Du Bois-Reymond [5] constructed an example of continuous function which trigonometric Fourier series diverges at some point. In 1923 A. N. Kolmogorov [11] proved that for a function from  $L^1(\mathbb{T})$  divergence of Fourier series can be everywhere. On the other hand according to Carleson-Hant theorem ([4], [7]) Fourier series of the functions from  $L^p(\mathbb{T})$ ,  $p > 1$ , converge a.e.. A natural question is whether the Fourier series of a function from  $L^p$  ( $p > 1$ ) or  $C$  may diverge on a given arbitrary set of measure zero. In fact the investigation of this problem began before Carleson's theorem. First S. B. Stechkin [14] in 1951 proved that for any set  $E \subset \mathbb{T}$  of measure zero there exists a function  $f \in L^2(0, 2\pi)$  which Fourier series diverges on  $E$ . Then in 1963 L. V. Taikov [15] proved  $f$  can be taken from  $L^p(0, 2\pi)$  for any  $1 \leq p < \infty$ . In 1965 Kahane and Katznelson [8] proved the existence of a continuous complex valued function which diverges on a given set of measure zero. Essentially developing Kahane-Katznelson approach V. V. Buzdin [3] proved that for any set of measure zero there exists a continuous real function which Fourier series diverges on that set. The same question is investigated also for the other classical orthonormal systems. Sh. V. Kheladze in [9] constructed a function from  $L^p(0, 1)$  ( $1 < p < \infty$ ) which Fourier-Walsh series diverges on a given set of measure zero. In another paper [10] he proved the same also for the Vilenkin systems. Then V. M. Bugadze [1] proved that for the Walsh system function in the mentioned theorem can be taken from  $L^\infty$ . In fact Bugadze proves the same also for Haar ([2]), Walsh-Paley and Walsh-Kachmaz systems([1]). Haar system in such problems is considered also in the papers M. A. Lunina [12] and V. I. Prochorenko [13]. Recently U. Goginava [6] proved that for any set of measure zero there exists a bounded function which Walsh-Fejer means diverges on that set. About other problems concerning the divergent Fourier series reader can find in the papers P. L. Ul'yanov [16] and W. L. Wade [17]. We have noticed this phenomena is common for general sequences of linear operators with localization

---

1991 *Mathematics Subject Classification.* 42A20.

*Key words and phrases.* general operators, localization property, divergence on a set, divergence of Fourier series.

property. We consider sequences of linear operators

$$(1) \quad U_n f(x) = \int_a^b K_n(x, t) f(t) dt, \quad n = 1, 2, \dots,$$

with

$$(2) \quad |K_n(x, t)| \leq M_n.$$

We say the sequence (1) has a localization property (*L*-property) if for any function  $f \in L^1(a, b)$  with  $f(x) = 1$  as  $x \in I = (\alpha, \beta) \subset [a, b]$  we have

$$\lim_{n \rightarrow \infty} U_n f(x) = 1 \quad \text{as } x \in I,$$

and the convergence is uniformly in each closed set  $A \subset I$ . We prove the following

**Theorem.** *If the sequence of operators (1) has a localization property, then for any set of measure zero  $E \subset [a, b]$  there exists a set  $G \subset [a, b]$  such that*

$$\liminf_{n \rightarrow \infty} U_n \mathbb{I}_G(x) \leq 0, \quad \limsup_{n \rightarrow \infty} U_n \mathbb{I}_G(x) \geq 1 \text{ for any } x \in E,$$

where  $\mathbb{I}_G(x)$  denotes the characteristic function of  $G$ .

The result of the theorem can be applied to the Fourier partial sums operators with respect to all classical orthogonal systems (trigonometric, Walsh, Haar, Franklin and Vilenkin systems). Moreover instead of partial sum we can discuss also linear means of partial sums corresponding to an arbitrary regular method  $T = \{a_{ij}\}$ . All these operators have localization property. So the following corollary is an immediate consequence of the main result.

**Corollary.** *Let  $\Phi = \{\phi_n(x), n \in \mathbb{N}\}$ ,  $x \in [a, b]$ , be one of the above mentioned orthogonal systems and  $T$  is an arbitrary regular linear method. Then for any set  $E$  of measure zero there exists a set  $G \subset [a, b]$  such that the Fourier series of its characteristic function  $f(x) = \mathbb{I}_G(x)$  with respect to  $\Phi$  diverges on  $E$  by  $T$ -method.*

**Remark.** *The function  $f(x)$  in the corollary can not be continuous in general. There are variety of sequences of Fourier operators which uniformly converge while  $f(x)$  is continuous.*

The following lemma gives a bound for the kernels of operators (1) if  $U_n$  has *L*-property.

**Lemma.** *If the sequence of operators  $U_n$  has *L*-property, then there exists a positive decreasing function  $\phi(u)$ ,  $u \in (0, +\infty)$ , such that if  $x \in [a, b]$  and  $n \in \mathbb{N}$  then*

$$(3) \quad |K_n(x, t)| \leq \phi(|x - t|) \text{ for almost all } t \in [a, b].$$

*Proof.* We define

$$\phi(u) = \sup_{n \in \mathbb{N}, x \in [a, b]} \text{essup}_{t:|t-x| \geq u} |K_n(x, t)|,$$

where  $\text{essup}_{t \in A} |g(t)|$  denotes  $\|g\|_{L^\infty(A)}$ . It is clear  $\phi(u)$  is decreasing and satisfies (3), provided  $\phi(u) < \infty$ ,  $u > 0$ . To prove  $\phi(u)$  is finite, let us suppose the converse, that is  $\phi(u_0) = \infty$  for some  $u_0 > 0$ . It means for any  $\gamma > 0$  there exist  $l_\gamma \in \mathbb{N}$  and  $c_\gamma \in [a, b]$  such that

$$(4) \quad |K_{l_\gamma}(c_\gamma, t)| > \gamma, \quad t \in E_\gamma \subset [a, b] \setminus (c_\gamma - u_0, c_\gamma + u_0), \quad |E_\gamma| > 0.$$

Consider the sequences  $c_k$  and  $l_k$  corresponding to the numbers  $\gamma_k = k$ ,  $k = 1, 2, \dots$ . We can fix an interval  $I$  with  $|I| = u_0/3$ , where the sequence  $\{c_k\}$  has infinitely many terms. Using this it can be supposed  $c_\gamma \in I$  in (4) and therefore we will have  $2I \subset (c_\gamma - u_0, c_\gamma + u_0)$ . So we can write

$$(5) \quad c_\gamma \in I, \quad E_\gamma \subset [a, b] \setminus 2I.$$

Thus we chose a sequence  $\gamma_k \nearrow \infty$  such that for corresponding sequences  $m_k = l_{\gamma_k}$ ,  $x_k = c_{\gamma_k}$  and  $E_k = E_{\gamma_k}$  we have

$$(6) \quad x_k \subset I, \quad E_k \subset (a, b) \setminus 2I,$$

$$(7) \quad |K_{m_k}(x_k, t)| \geq k^3, \quad t \in E_k,$$

$$(8) \quad \sup_{1 \leq i < k} |U_{m_k} \mathbb{I}_{E_i}(x)| < 1, \quad x \in I,$$

$$(9) \quad |E_k| \cdot \max_{1 \leq i < k} M_{m_i} < 1, (k > 1).$$

We do it by induction. Taking  $\gamma_1 = 1$  we will get  $m_1$  satisfying (7). This follows from (4). Now suppose we have already chosen the numbers  $\gamma_k$  and  $m_k$  satisfying (6)-(9) for  $k = 1, 2, \dots, p$ . According to  $L$ -property  $U_n \mathbb{I}_{E_i}(x)$  converge to 0 uniformly in  $I$  for any  $i = 1, 2, \dots, p$ . On the other hand because of (2) and (4) from  $\gamma \rightarrow \infty$  follows  $l_\gamma \rightarrow \infty$ . Hence we can chose a number  $\gamma_{p+1} > (p+1)^3$  such that the corresponding  $m_{p+1}$  satisfies the inequality

$$(10) \quad |U_{m_{p+1}} \mathbb{I}_{E_i}(x)| < 1, \quad x \in I, \quad i = 1, 2, \dots, p.$$

This gives (8) in the case  $k = p+1$ . According to (4) and the bound  $\gamma_{p+1} > (p+1)^3$  we will have also (7). Finally, since  $E_{p+1}$  may have enough small measure we can guarantee (9) for  $k = p+1$ . So the construction of the sequence  $\gamma_k$  with (6)-(9) is complete. Now consider the function

$$(11) \quad g(x) = \sum_{i=1}^{\infty} \frac{\mathbb{I}_{E_i}(x)}{k^2}.$$

We have  $g \in L^1$  and  $\text{supp } g \subset [a, b] \setminus (2I)$ . Since  $x_k \in I$ , using the relations (6)-(9), we obtain

$$\begin{aligned} |U_{m_k} g(x_k)| &\geq \\ &\geq \frac{|U_{m_k} \mathbb{I}_{E_k}(x_k)|}{k^2} - \sum_{i=1}^{k-1} \frac{|U_{m_k} \mathbb{I}_{E_i}(x_k)|}{i^2} - \sum_{i=k+1}^{\infty} \frac{|U_{m_k} \mathbb{I}_{E_i}(x_k)|}{i^2} \\ &\geq k - \sum_{i=1}^{k-1} \frac{1}{i^2} - M_{m_k} \sum_{i=k+1}^{\infty} \frac{|E_i|}{i^2} \geq k - 2. \end{aligned}$$

This is a contradiction, because the convergence  $U_n g(x) \rightarrow 0$  is uniformly on  $I$  according to  $L$ -property.  $\square$

We say a family  $\mathcal{I}$  of mutually disjoint semi-open intervals is a regular partition for an open set  $G \subset (a, b)$  if  $G = \cup_{I \in \mathcal{I}} I$  and each interval  $I \in \mathcal{I}$  has two adjacent intervals  $I^+, I^- \in \mathcal{I}$  with

$$(12) \quad 2I \subset I^* = I \cup I^+ \cup I^-.$$

It is clear any open set has a regular partition.

*Proof of Theorem.* For a given set  $E$  of measure zero we will construct a definite sequence of open sets  $G_k, k = 1, 2, \dots$ , with regular partitions  $\mathcal{I}_k, k = 1, 2, \dots$ . They will satisfy the conditions

- 1) if  $I \in \mathcal{I}_k$  and  $I = [\alpha, \beta)$  then  $\alpha, \beta \notin E$ ,
- 2) if  $I, J \in \cup_{j=1}^k \mathcal{I}_j$  then  $J \cap I \in \{\emptyset, I, J\}$ ,
- 3)  $E \subset G_k \subset G_{k-1}, (G_0 = [a, b])$ .

In addition for any interval  $I \in \mathcal{I}$  we fix a number  $\nu(I) \in \mathbb{N}$  such that

- 4) if  $I, J \in \cup_{j=1}^k \mathcal{I}_j$  and  $I \subset J$  then  $\nu(I) \geq \nu(J)$ ,
- 5)  $\sup_{x \in I} |U_{\nu(I)} \mathbb{I}_{G_l}(x) - 1| < 1/k^2$ , if  $I \in \mathcal{I}_k$  and  $l \leq k$ ,
- 6)  $\sup_{x \in I} |U_{\nu(I)} \mathbb{I}_{G_k}(x)| < 1/k^2$ , if  $I \in \mathcal{I}_l$  and  $l < k$ .

We define  $G_1$  and its partition  $\mathcal{I}_1$  arbitrarily, just ensuring the condition 1). It may be done because  $|E| = 0$  and so  $E^c$  is everywhere dense on  $[a, b]$ . Then using  $L$ -property for any interval  $I \in \mathcal{I}_1$  we can find a number  $\nu(I) \in \mathbb{N}$  satisfying 5) for  $k = 1$ . Now suppose we have already chosen  $G_k$  and  $\mathcal{I}_k$  with the conditions 1)-6) for all  $k \leq p$ . Obviously we can chose an open set  $G_{p+1}, E \subset G_{p+1} \subset G_p$ , satisfying 1), 2) and the bound

$$|G_{p+1} \cap I| < \delta(I), \quad I \in \cup_{k=1}^p \mathcal{I}_k,$$

where

$$\delta(I) = \frac{1}{6(p+1)^2 \max \left\{ M_{\nu(I)}, M_{\nu(I^+)}, M_{\nu(I^-)}, \frac{\phi(|I|/2)}{|I|} \right\}},$$

and the function  $\phi(u)$  is taken from the lemma. Suppose  $I \in \mathcal{I}_l$  and  $l < p+1$ . We have

$$(13) \quad |U_{\nu(I)} \mathbb{I}_{G_{p+1}}(x)| \leq |U_{\nu(I)} \mathbb{I}_{G_{p+1} \cap I^*}(x)| + |U_{\nu(I)} \mathbb{I}_{G_{p+1} \cap (I^*)^c}(x)|.$$

Using the lemma and the bound

$$\delta(J) \leq \frac{|J|}{6\phi(|J|/2)(p+1)^2}, \quad J \in \mathcal{I}_l,$$

for any  $x \in I$  we get

$$(14) \quad \begin{aligned} & |U_{\nu(I)} \mathbb{I}_{G_{p+1} \cap (I^*)^c}(x)| \\ & \leq \sum_{J \in \mathcal{I}_l: J \neq I, I^+, I^-} \int_{G_{p+1} \cap J} \phi(|x - t|) dt \\ & \leq \sum_{J \in \mathcal{I}_l: J \neq I, I^+, I^-} \int_{G_{p+1} \cap J} \phi\left(\frac{|J|}{2}\right) dt \\ & \leq \sum_{J \in \mathcal{I}_l: J \neq I, I^+, I^-} |G_{p+1} \cap J| \phi\left(\frac{|J|}{2}\right) \\ & \leq \sum_{J \in \mathcal{I}_l: J \neq I, I^+, I^-} \delta(J) \phi\left(\frac{|J|}{2}\right) \\ & \leq \frac{1}{6(p+1)^2} \sum_{J \in \mathcal{I}_l: J \neq I, I^+, I^-} |J| < \frac{1}{6(p+1)^2}, \quad x \in I. \end{aligned}$$

On the other hand we have

$$\delta(I), \delta(I^+), \delta(I^-) \leq \frac{1}{6 \cdot (p+1)^2 M_{\nu(I)}},$$

and therefore

$$(15) \quad \begin{aligned} |U_{\nu(I)} \mathbb{I}_{G_{p+1} \cap I^*}(x)| &\leq M_{\nu(I)} |G_{p+1} \cap I^*| \\ &\leq M_{\nu(I)} (\delta(I) + \delta(I^+) + \delta(I^-)) \leq \frac{1}{2(p+1)^2}, \quad x \in [a, b]. \end{aligned}$$

Combining (13), (14) and (15) we get 6) in the case  $k = p + 1$ . Now we chose the partition  $\mathcal{I}_{p+1}$  satisfying just conditions 1) and 2). Using  $L$  property we may define numbers  $\nu(I)$  for  $I \in \mathcal{I}_{p+1}$  satisfying the condition 5) with  $k = p + 1$ . Hence the construction of the sets  $G_k$  is complete. Now denote

$$G = \bigcup_{i=1}^{\infty} (G_{2i-1} \setminus G_{2i}),$$

we have

$$U_n \mathbb{I}_G(x) = \sum_{k=1}^{\infty} (-1)^{k+1} U_n \mathbb{I}_{G_k}(x).$$

For any  $x \in E$  there exists a unique sequence  $I_1 \supset I_2 \supset \dots \supset I_k \supset \dots$ ,  $I_k \in \mathcal{I}_k$ , such that  $x \in I_k$ ,  $k = 1, 2, \dots$ . According to 6) for  $l > k$  we have

$$|U_{\nu(I_k)} \mathbb{I}_{G_l}(x)| \leq \frac{1}{l^2}, \quad l > k.$$

From 5) it follows that

$$|U_{\nu(I_k)} \mathbb{I}_{G_l}(x) - 1| \leq \frac{1}{k^2}, \quad l \leq k.$$

Thus we obtain

$$\begin{aligned} &|U_{\nu(I_k)} \mathbb{I}_G(x) - \sum_{l=1}^k (-1)^{l+1}| \\ &\leq \sum_{l=1}^k |U_{\nu(I_k)} \mathbb{I}_{G_l}(x) - 1| + \sum_{l=k+1}^{\infty} |U_{\nu(I_k)} \mathbb{I}_{G_l}(x)| \\ &\leq k \cdot \frac{1}{k^2} + \sum_{l=k+1}^{\infty} \frac{1}{l^2} < \frac{2}{k} \end{aligned}$$

Since the sum  $\sum_{i=1}^k (-1)^{k+1}$  takes values 0 and 1 alternately we get

$$\lim_{t \rightarrow \infty} U_{\nu(I_{2t})} \mathbb{I}_G(x) = 0, \quad \lim_{t \rightarrow \infty} U_{\nu(I_{2t+1})} \mathbb{I}_G(x) = 1$$

for any  $x \in E$ . The proof of theorem is complete.  $\square$

#### REFERENCES

- [1] Bugadze, V. M. Divergence of Fourier-Walsh series of bounded functions on sets of measure zero. (Russian) Mat. Sb. 185 (1994), no. 7, 119–127
- [2] Bugadze, V. M. On the divergence of Fourier-Haar series of bounded functions on sets of measure zero. (Russian) Mat. Zametki 51 (1992), no. 5, 20–26.
- [3] Buzdalov, V. V. Unboundedly diverging trigonometric Fourier series of continuous functions. (Russian) Mat. Zametki 7 1970 7–18.
- [4] Carleson L., On convergence and growth of partial sums of Fourier series, Acta Math., 1966, 116, 135–157,
- [5] Du Bois-Reymond T. , Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungsformen, Abh. Akad. Wiss. München, X (1876), 1 103.

- [6] Goginava U., On devergence of Walsh-Fejer means of bounded functions on sets of measure zero. *Acta Math. Hungarica* 121 (3) (2008), 359-369
- [7] Hunt R.A., On the convergence of Fourier series in orthogonal expansions and their continuous analogues, Southern Illinois University Press, 1968, 235–256,
- [8] Kahane J-P., Katznelson Y., Sur les ensembles de divergence des series trigonométriques, *Studia math.*, XXVI (1966), 305306
- [9] Kheladze Sh. V., On everywhere devergence of Fourier series by bounded type Vilenkin systems, *Trudi Tbil. Math. Ins. Gruz. SSR*, 1978, v. 58, 225-242.
- [10] Kheladze Sh. V., On everywhere devergence of Fourier-Walsh series, *Soobsh. Geargian ANS*, 1975, v.77, No 2, 305-307.
- [11] Kolmogoroff A. N. , Une serie de Fourier-Lebesgue divergente partout, *C. r. Acad. Sci., Paris*, 183 (1926), 13271328.
- [12] Lunina M. A., On the sets of unbounded devergence of series with respect to Haar system, *Vestnik MGU, math.*, 1976, No 4, 13-20.
- [13] Prochorenko V. I., Divergent series with respect to Haar system, *Izvestia Vuzov, Math.*, 1971, No 1, 62-68.
- [14] Stechkin S. B., On the convergence and devergence of trigonometric series, *Uspechi Math. Nauk*, VI, vol. 2(42), 1951, 148-149 (in Russian).
- [15] Taikov L. V., On the devergence of Fourier series by rearranged trigonometric system, *Uspechi Math. Nauk*, XVIII, v. 5(113), 1963, 191-198 (in Russian).
- [16] Ul'yanov P. L., A. N. Kolmogorov and divergent Fourier series, *Jour. Russ. Math. Surv.* 1983, vol. 38, No.4, 57-100;
- [17] Wade W. R., Resent development in the theory of Walsh series, *Internat. Journ. of Math. and Math. Sci.*, 1982, vol. 5, No 4, 625-673.

INSTITUTE OF MATHEMATICS OF ARMENIAN NATIONAL ACADEMY OF SCIENCES, BAGHRAMIAN  
AVE.- 24B, 375019, YEREVAN, ARMENIA

*E-mail address:* g.karagulyan@yahoo.com